Volume Title: Annals of Economic and Social Measurement, Volume 2, number 4

Volume Author/Editor: Sanford V. Berg, editor

Volume Publisher: NBER

Volume URL: http://www.nber.org/books/aesm73-4

Publication Date: October 1973

Chapter Title: A Test for Systematic Variation in Regression Coefficients

Chapter Author: David A. Belsley

Chapter URL: http://www.nber.org/chapters/c9940

Chapter pages in book: (p. 495 - 499)

Annals of Economic and Social Measurement, 2/4, 1973

A TEST FOR SYSTEMATIC VARIATION IN REGRESSION COEFFICIENTS

BY DAVID A. BELSLEY*

This paper offers a statistical test of the constancy of the parameters of a linear regression. The F test is based on transformed residuals which result from OLS applied to the given equation under the null hypothesis of constancy.

SOME NOTATION

We consider the model

$$y(t) = x'(t)\beta(t) + \varepsilon(t) \tag{1}$$
$$\beta(t) = \Gamma z(t) + u(t)$$

where

$x(t)$, $z(t)$ K and R vectors, respectively,
$\varepsilon(t)$ spherically distributed with $E\varepsilon\varepsilon' = \sigma^2 I$,
$u(t)$ independent over time with $Euu' = \sigma_u^2\Omega$.

(See preceding article for motivation.

In what follows we consider the special case $\sigma_u^2 = 0$, i.e., variation in $\beta(t)$ is systematic and non random. Hence, we may write

$$\begin{aligned} y(t) &= x'(t)\Gamma z(t) + \varepsilon(t) \qquad \Gamma = [\gamma_1 \cdots \gamma_R] \\ &= [x'(t) \otimes z'(t)]\Lambda + \varepsilon(t) \end{aligned} \tag{2}$$

where

$$\Lambda = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_R \end{bmatrix}.$$

Let

$$Y = [y(t)], \qquad \underset{T \times K}{X = \begin{bmatrix} x'(1) \\ \vdots \\ x'(T) \end{bmatrix}}, \qquad \underset{T \times R}{Z = \begin{bmatrix} z'(1) \\ \vdots \\ z'(T) \end{bmatrix}} \qquad \underset{T \times KR}{D = \begin{bmatrix} x'(1) \otimes z'(1) \\ \vdots \\ x'(T) \otimes z'(T) \end{bmatrix}}.$$

Then (2) becomes

$$Y = D\Lambda + \varepsilon \tag{3}$$

* Research supported by National Science Foundation Grant GJ-1154x to the National Bureau of Economic Research, Inc. Research Report W0006. This report has not undergone the full critical review accorded the National Bureau's studies, including review by the Board of Directors.

and we note that we may write

$$D = [\mathscr{Z}_1 \mathscr{Z}_2 \ldots \mathscr{Z}_R][X \otimes I], \tag{4}$$

where $\mathscr{Z}_r = \operatorname{diag} Z_r$ and Z_r is the rth column of Z.

Thus, (3) becomes

$$Y = \sum_{r=1}^{R} \mathscr{Z}_r X \gamma_r + \varepsilon. \tag{5}$$

Remarks

Our purpose here is to determine a test of the null hypothesis that $\beta(t) = \beta$, i.e., is constant, for all t. Clearly a regression could be run on (3) directly if the z's were known, but alternative modeling tests would be cumbersome given the size of $(D'D)^{-1}$ even for moderate K and R.

In what follows a two-step test is determined that looks to be efficient and does not require inversion of $D'D$. Alternative Z matrices may be compared with a minimum of computation. The first step is OLS of Y on X without regard to Z. The second step consists of regressing a transformed set of residuals from step one on the similarly transformed z's. H_0 may be tested with the results of the second regression.

Step One: OLS Y on X

First regress Y on X to get

$$\begin{aligned} b &= (X'X)^{-1}X'Y \\ &= (X'X)^{-1}X'D\Lambda + (X'X)^{-1}X'\varepsilon \\ &= (X'X)^{-1}X'\sum_r \mathscr{Z}_r X\gamma_r + (X'X)^{-1}X'\varepsilon \end{aligned} \tag{6}$$

and

$$\begin{aligned} e &= Y - Xb = HY \qquad (H = I - X(X'X)^{-1}X') \\ &= H(D\Lambda + \varepsilon) \\ &= [H\mathscr{Z}_1 X \ldots H\mathscr{Z}_R X]\Lambda + H\varepsilon \\ &\equiv [V_1 \ldots V_R]\Lambda + H\varepsilon \\ &= \sum_{r=1}^{R} V_r\gamma_r + H\varepsilon \end{aligned}$$

where the V_r are the residual matrices from an auxiliary regression of $\mathscr{Z}_r X$ on X.

This regression need not be run in practice. The relevance of V_r is seen from

$$H\mathscr{Z}_r X_r = \mathscr{Z}_r X - X(X'X)^{-1}X'\mathscr{Z}_r X = \mathscr{Z}_r X - XB_r \equiv V_r.$$

where B_r is the set of regression coefficients from $\mathscr{Z}_r X = XB_r + V_r$.[1]

Thus we have

$$e = \Sigma V_r \gamma_r + H\varepsilon. \tag{8}$$

We recall that H is idempotent, has rank $T - K$, and hence there exists an orthogonal C such that $C'HC = \begin{bmatrix} I_{T-K} & 0 \\ 0 & 0 \end{bmatrix} \equiv G$. Further we note $HV_r = V_r$, $r = 1 \ldots R$ and $He = e$. Hence, we may write

$$C'HCC'e = C'HCC'\Sigma V_r \gamma_r + C'HCC'\varepsilon \tag{9}$$

or

$$GC'e = GC'\Sigma V_r \gamma_r + GC'\varepsilon$$

and, partitioning $C = [C_1 C_2]$ so that the first $T - K$ rows of (9) become

$$f \equiv C_1'e = C_1' \sum_{r=1}^{R} V_r \gamma_r + C_1'\varepsilon \tag{10}$$

$$= C_1' \sum_{r=2}^{R} \mathscr{Z}_r X \gamma_r + \eta.^2$$

This last inequality comes from noting that $V_r = H\mathscr{Z}_r X$, and hence $C'V_r = C'H\mathscr{Z}_r X = C'HCC'\mathscr{Z}_r X = GC'\mathscr{Z}_r X$, which implies $C_1'V_r = C_1'\mathscr{Z}_r X$. We have also let $C_1\varepsilon \equiv \eta$.

We also note that η is spherically distributed, since $E\eta = 0$, $V\eta = E\eta\eta' = EC_1'\varepsilon\varepsilon' C_1 = \sigma_\varepsilon^2 C_1' C_1 = \sigma_\varepsilon^2 I_{T-K}$, due to the orthogonality of C.

It is the transformed residuals $f = C_1'e$ that we make use of in step two. The transformation C_1' comes from finding an orthogonal set of eigenvectors of $H = I - X(X'X)^{-1}X'$, and hence f depends only on knowledge of X and Y and does not require knowledge of Z.

Step Two

It is clear from (10) that the residuals from step one depend in a very involved way on the interrelation of X and Z through the terms $\mathscr{Z}_r X$. However, under the null hypothesis $H_0 : \beta(t) = \beta$, these terms disappear, and a simpler test is available.

Consider a mechanical regression of f on Z transformed by C_1' (which depends only on X):

$$f = C_1' Z\delta + \psi. \tag{11}$$

[1] In passing we note from (6) that

$$b = \Sigma(X'X)^{-1}X'\mathscr{Z}_r X\gamma_r + (X'X)^{-1}X'\varepsilon$$
$$= \Sigma B_r \gamma_r + (X'X)^{-1}X'\varepsilon.$$

Hence, $Eb = \Sigma B_r \gamma_r$, a weighted sum of the γ_r, and $V(b) = \sigma^2(X'X)^{-1}$.

[2] This latter sum goes from $r = 2$ to R since, if Z_1 (the first col. of Z) is a column vector of all ones, then $\mathscr{Z}_1 = I$ and hence $V_1 \equiv \mathscr{Z}_1 X - XB_1 = X - XB_1$, the least squares residuals of the auxiliary equation $X = XB_1 + V_1$. These residuals must necessarily be zero, since $B_1 = I$ does the trick of minimizing the sum of squares. Hence, $C_1'V_1 = 0 = C_1'\mathscr{Z}_1 X = C_1'X$.

OLS gives

$$(12) \qquad d = (Z'C_1C_1'Z)^{-1}Z'C_1f \qquad \text{and from (10)}$$

$$= (Z'C_1C_1'Z)^{-1}Z'C_1C_1'\Sigma\mathscr{Z}_rX\gamma_r + (Z'C_1C_1'Z)^{-1}Z'C_1C_1'\varepsilon$$

$$\equiv (Z'QZ)^{-1}Z'Q\sum_{r=2}^{R}\mathscr{Z}_rX\gamma_r + (Z'QZ)^{-1}Z'Q\varepsilon$$

where $Q \equiv C_1C_1'$.

Under the null hypothesis $H_0: \beta(t) = \beta$, $\gamma_r = 0$ for $r = 2 \ldots R$, and hence the first term of (12) is 0. That is, under H_0:

$$(13) \qquad d = (Z'QZ)^{-1}Z'Q\varepsilon$$

$$= (Z'QZ)^{-1}Z'C_1f.$$

In addition, from (10) we have under H_0 that

$$(14) \qquad f = C_1'\varepsilon.$$

Further, we note for future reference that Q is idempotent—since $QQ = C_1C_1'C_1C' = C_1IC_1' = C_1C_1' = Q$—and of rank $T - K$.

Now consider the residuals g of this second step; using (13) and (14),

$$(15) \qquad g \equiv f - C_1'Zd$$

$$= C_1'\varepsilon - C_1'Z(Z'QZ)^{-1}Z'Q\varepsilon$$

$$= C_1'[I - Z(Z'QZ)^{-1}Z'Q]\varepsilon$$

$$\equiv N\varepsilon \qquad \text{where we let } N = C_1'[I - Z(Z'QZ)^{-1}Z'Q].$$

Now

$$g'g = \varepsilon'N'N\varepsilon$$

$$= \varepsilon'[I - QZ(Z'QZ)^{-1}Z']C_1C_1'[I - Z(Z'QZ)^{-1}Z'Q]\varepsilon$$

$$= \varepsilon'[Q - QZ(Z'QZ)^{-1}Z'Q][Q - QZ(Z'QZ)^{-1}Z'Q]\varepsilon$$

$$(16) \qquad \equiv \varepsilon'MM\varepsilon \qquad \text{where } M \equiv Q - QZ(Z'QZ)^{-1}Z'Q$$

$$\equiv \varepsilon M\varepsilon$$

since M is seen to be idempotent with $\rho(M) = \operatorname{tr} M = T - K - R$. And hence,

$$(17) \qquad g'g \leftrightarrow \sigma_\varepsilon^2 X^2_{T-K-R}.$$

From (13) we have

$$(18) \qquad \mathrm{d} = (Z'QZ)^{-1}Z'Q\varepsilon \equiv B\varepsilon$$

and

$$
\begin{aligned}
BM &= (Z'QZ)^{-1}Z'Q[Q - QZ(Z'QZ)^{-1}Z'Q] \\
&= (Z'QZ)^{-1}Z'Q - (Z'QZ)^{-1}Z'Q = 0.
\end{aligned}
$$

Hence, the linear form (18) is distributed independently of the quadratic form (17) and the usual tests of significance on d may take place. Under H_0: E$d = 0$, and hence a t value for a specific d at $T - K - R$ degrees of freedom in excess of the test level rejects the null hypothesis.

Boston College, and
National Bureau of Economic Research